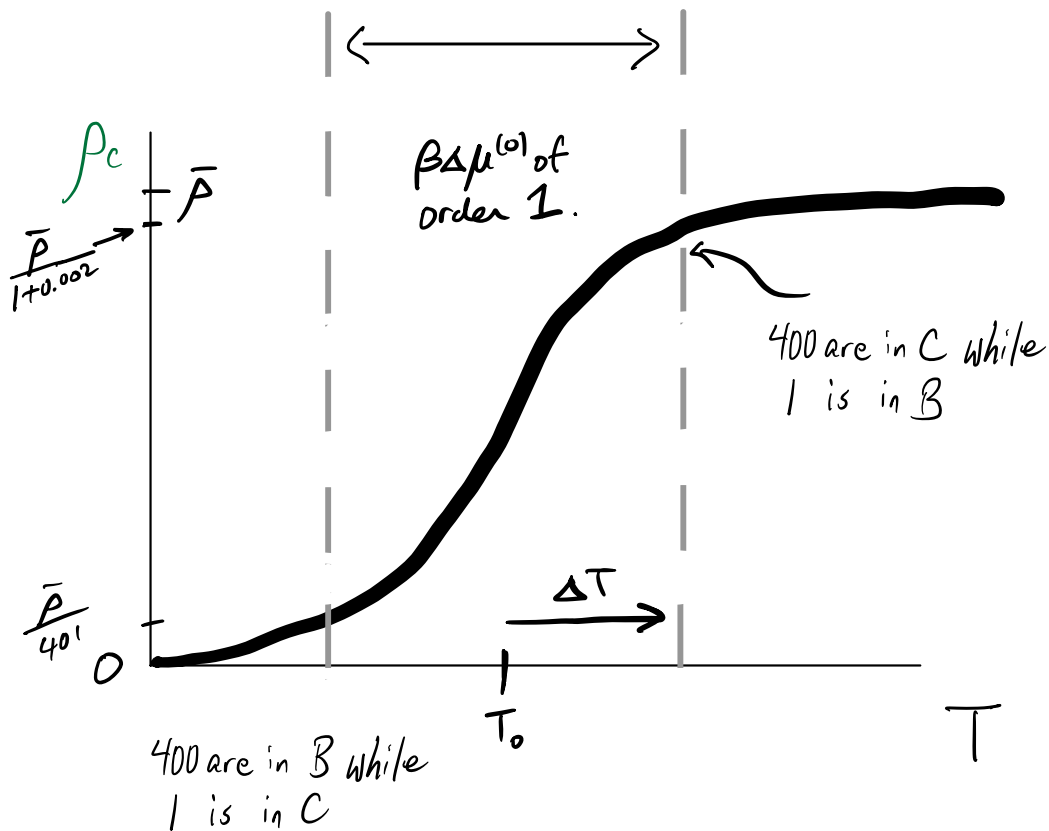


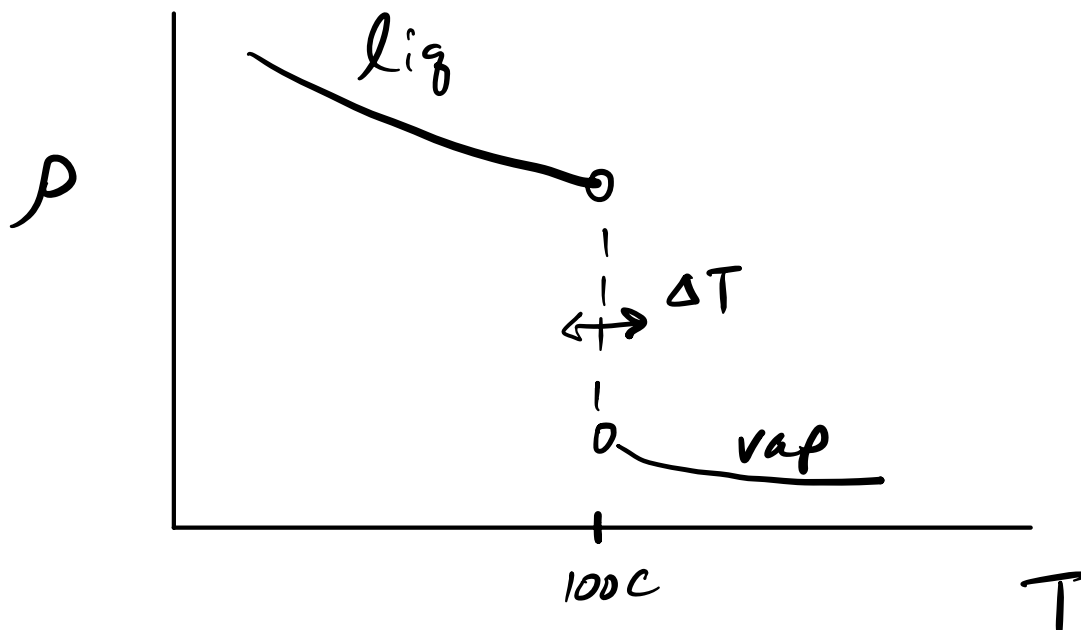
Lecture 16

Recall from last lecture...

How broad is this crossover? $\Delta T \approx$ a few degrees C.



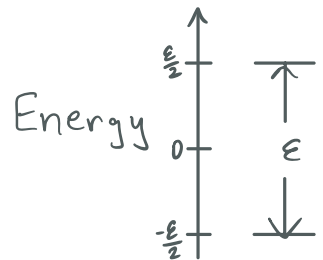
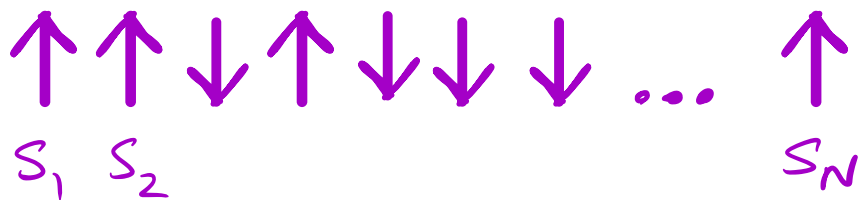
By contrast, for a liquid-vapor transition at fixed p ,
How broad? $\Delta T \approx 0$ Not broad at all!



Phase transitions arise due to interactions.

Recall (Lecture 9)

Non-interacting spins in an external field.



$$S_i = \begin{cases} +1, & \text{aligned with the field} \\ -1, & \text{anti-aligned} \end{cases}$$

$$\nu = \{S_1, S_2, \dots, S_N\} \quad E(\nu) = -\frac{\epsilon}{2} \sum_i S_i$$

We are after a relationship between T and the fraction of excited spins
(an excited spin has $S_i = -1$)

$$P(S_1, S_2, \dots, S_N) \propto e^{-\beta E(S_1, S_2, \dots, S_N)} = e^{\beta \frac{\epsilon}{2} \sum_i S_i}$$

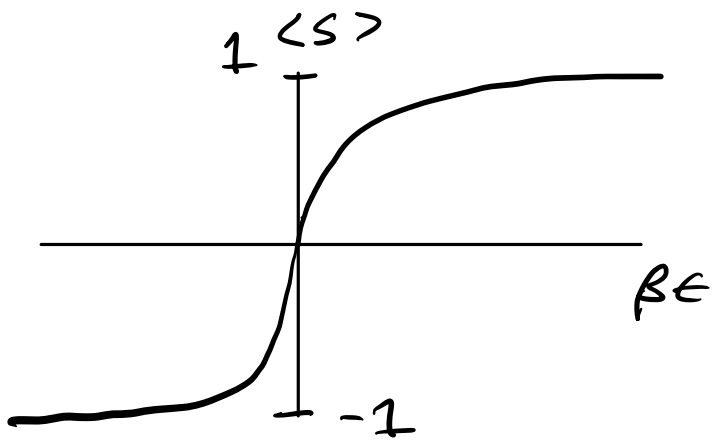
It factorizes! $\Rightarrow$ $= e^{\beta \frac{\epsilon}{2} S_1} e^{\beta \frac{\epsilon}{2} S_2} \dots e^{\beta \frac{\epsilon}{2} S_N}$

$$= p(S_1) p(S_2) \dots p(S_N) \leftarrow \begin{array}{l} \text{Spins are} \\ \text{statistically} \\ \text{independent} \end{array} \quad \left(\begin{array}{l} \text{b/c they are} \\ \text{non-interacting} \end{array} \right)$$

Using this, we found...

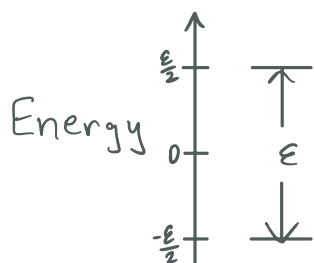
$$\begin{aligned}\langle S \rangle &= p(S=1) * 1 + p(S=-1) * (-1) \\ &= \frac{e^{\beta E/2} - e^{-\beta E/2}}{e^{\beta E/2} + e^{-\beta E/2}} = \tanh\left(\frac{\beta E}{2}\right)\end{aligned}$$

$$\Rightarrow \langle S \rangle = \frac{e^{\beta E} - 1}{e^{\beta E} + 1} (*)$$



A smooth function of βE

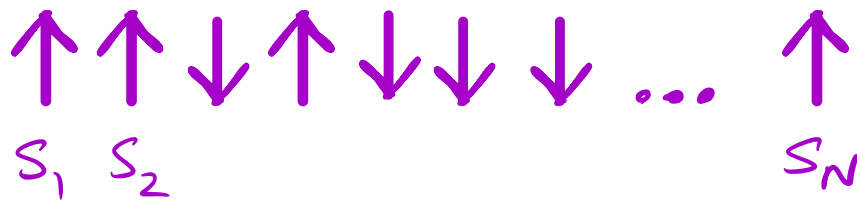
How would you change E in an experiment?



Tune the field strength H
which would then change E

$$E(v) = -\mu_0 H \sum_i S_i$$

$h = \mu \cdot H$
 magnetic moment
 field strength



Actual spins tend to align with their neighbors
 (QM has a role here - exchange interaction)

Ising Hamiltonian

non-interacting

an interaction
 between spins

$$E(v) = -h \sum_i S_i - \frac{J}{2} \sum_{i,j} S_i S_j$$

"Coupling constant"

If aligned, $S_i S_j = 1$
 If not, $S_i S_j = -1$

The $1/2$ allows us to
 include $i=1, j=2$
 and $i=2, j=1$ terms
 in the sum - "double counting"

' is shorthand that means we
 only include nearest neighbor $i \leftrightarrow j$.

Note

$$E|_{h=0} = -\frac{J}{2} \sum_{i,j}' S_i S_j$$

has up-down symmetry

↑ ↑ ↓ ↑ ↓ ↓ ↓ ...
 $S_1 S_2 \quad S_N$

$E(\quad) = E_{\text{top}}$

$E(\quad) = E_{\text{bottom}}$

↓ ↓ ↑ ↓ ↑ ↑ ↑ ... ↓
 $S_1 S_2 \quad S_N$

$$E_{\text{top}} = E_{\text{bottom}}$$

In thermal equilibrium, the probability of a microstate comes from the Boltzmann distribution (as always)

$$P(\nu) = \frac{e^{\beta h \sum_i S_i + \beta \frac{J}{2} \sum_{i,j}' S_i S_j}}{Q}$$

where

$$Q = \sum_{\nu} e^{\beta h \sum_i S_i + \beta \frac{J}{2} \sum_{i,j}' S_i S_j}$$

$$= \sum_{S_1=\pm 1} \sum_{S_2=\pm 1} \dots \sum_{S_N=\pm 1} e^{\beta h \sum_i S_i + \beta \frac{J}{2} \sum_{i,j} S_i S_j}$$

When $J=0$, we're back to the non-interacting spins!

Factorized probability distribution

$$g = e^{-\beta E_{\text{up}}} + e^{-\beta E_{\text{down}}} \\ e^{-\beta h} + e^{\beta h}$$

$$Q = \underbrace{\sum_{S_1=\pm 1} e^{\beta h S_1}}_g \underbrace{\sum_{S_2=\pm 1} e^{\beta h S_2}}_g \dots \underbrace{\sum_{S_N=\pm 1} e^{\beta h S_N}}_g = g^N \\ = (e^{-\beta h} + e^{\beta h})^N$$

But when $J \neq 0$, things are not so easy.

The distribution doesn't factorize, so I must sum over the 2^N microstates, not just the microstates for a single spin.

We are ^{still} after a relationship between T and the fraction of ~~excited~~ ^{up} spins ^{and h}

Net Magnetization: $M = \sum_{\text{spins } i} S_i$

Net Magnetization per site: $m = \frac{M}{N}$

If we could explicitly compute $Q(\beta h, \beta J) \dots$

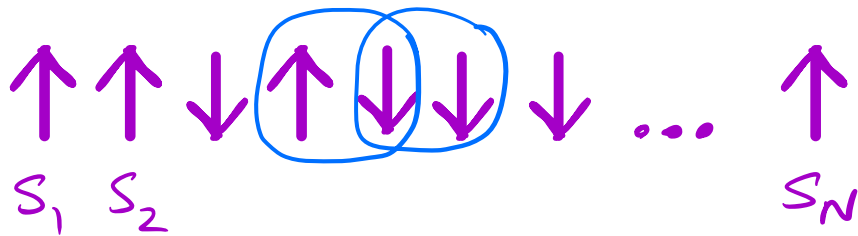
$$-\frac{\partial \ln Q}{\partial \beta} = \langle E \rangle \quad \frac{\partial \ln Q}{\partial (\beta h)} = \langle M \rangle$$

$$Q = \sum_{\nu} e^{\beta h \sum_i S_i + \frac{\beta J}{2} \sum'_{i,j} S_i S_j}$$

$$\begin{aligned} \frac{\partial \ln Q}{\partial (\beta h)} &= \frac{1}{Q} \frac{\partial Q}{\partial (\beta h)} = \frac{1}{Q} \sum_{\nu} \left(\sum_i S_i \right) e^{\beta h \sum_i S_i + \frac{\beta J}{2} \sum'_{i,j} S_i S_j} \\ &= \sum_{\nu} M(\nu) P(\nu) = \langle M \rangle \end{aligned}$$

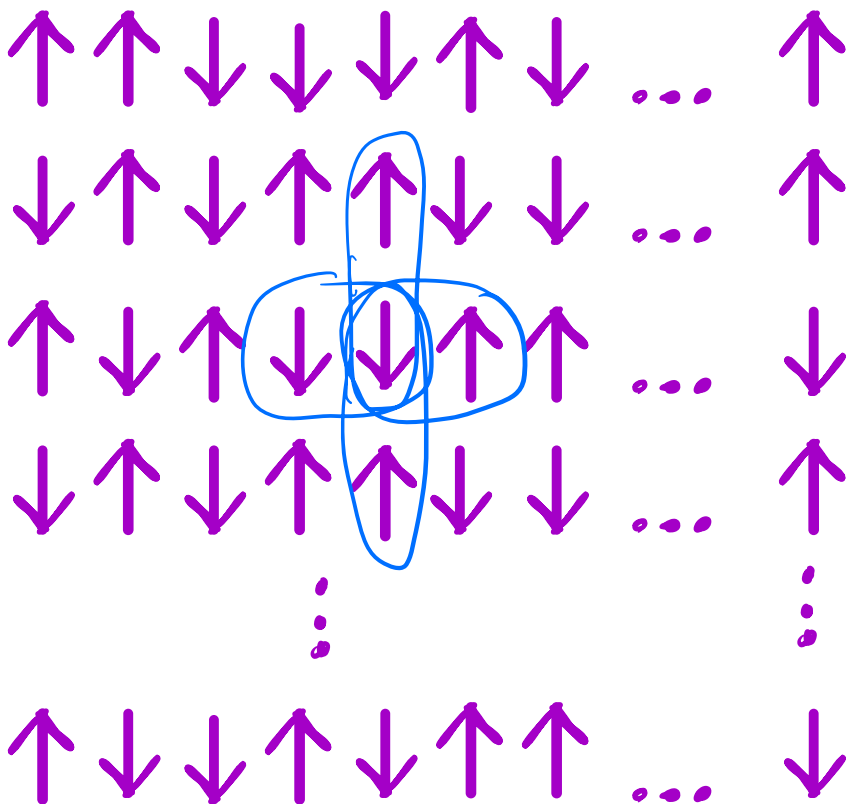
Well can we explicitly compute $Q(\beta h, \beta J)$? *It depends.*

1D...



Yup. You can do it. Transfer matrices are fun!

2D...



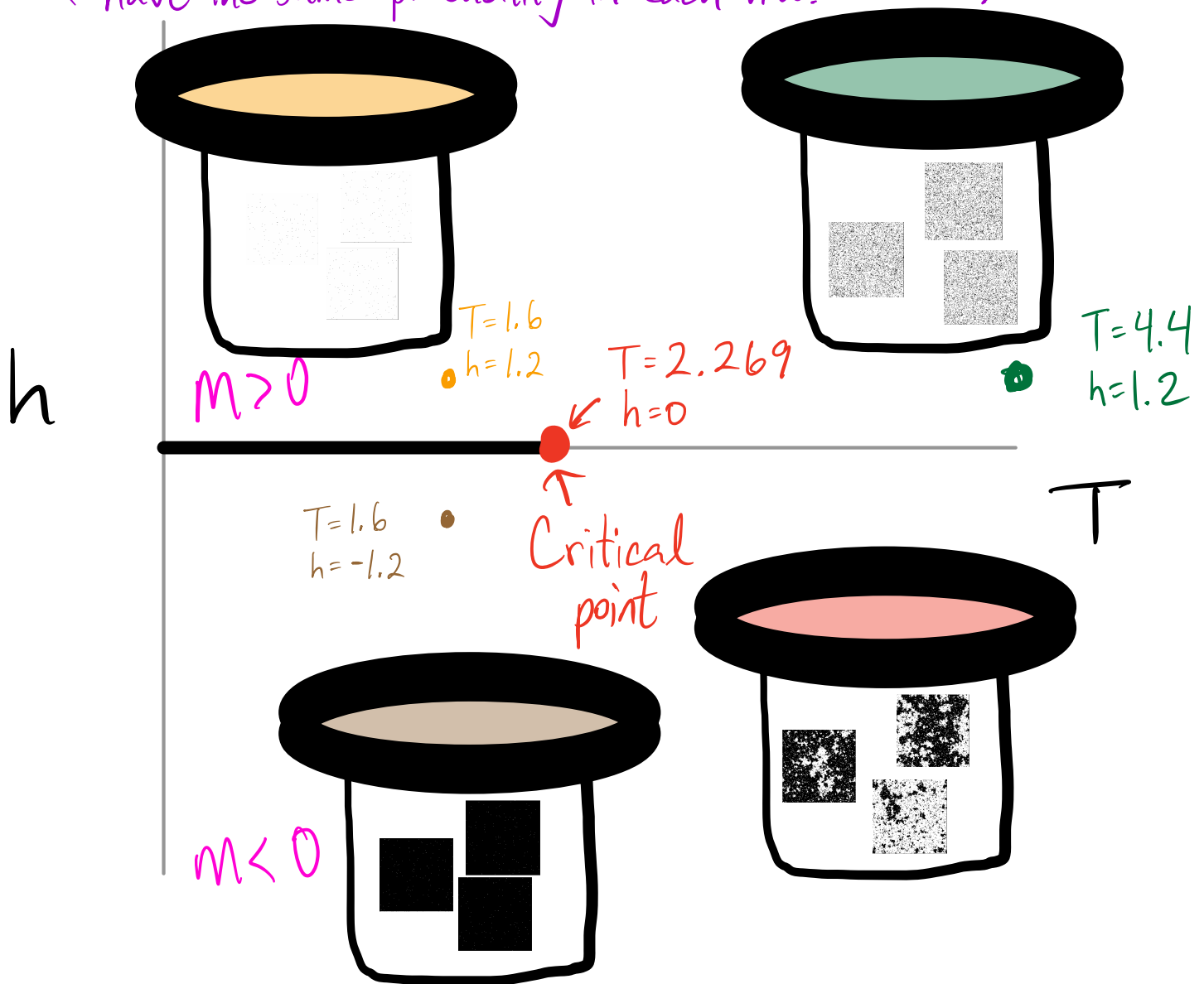
One can do it
by hand. Perhaps
not you or I.
Lars Onsager
did it.

> 2D... Nope

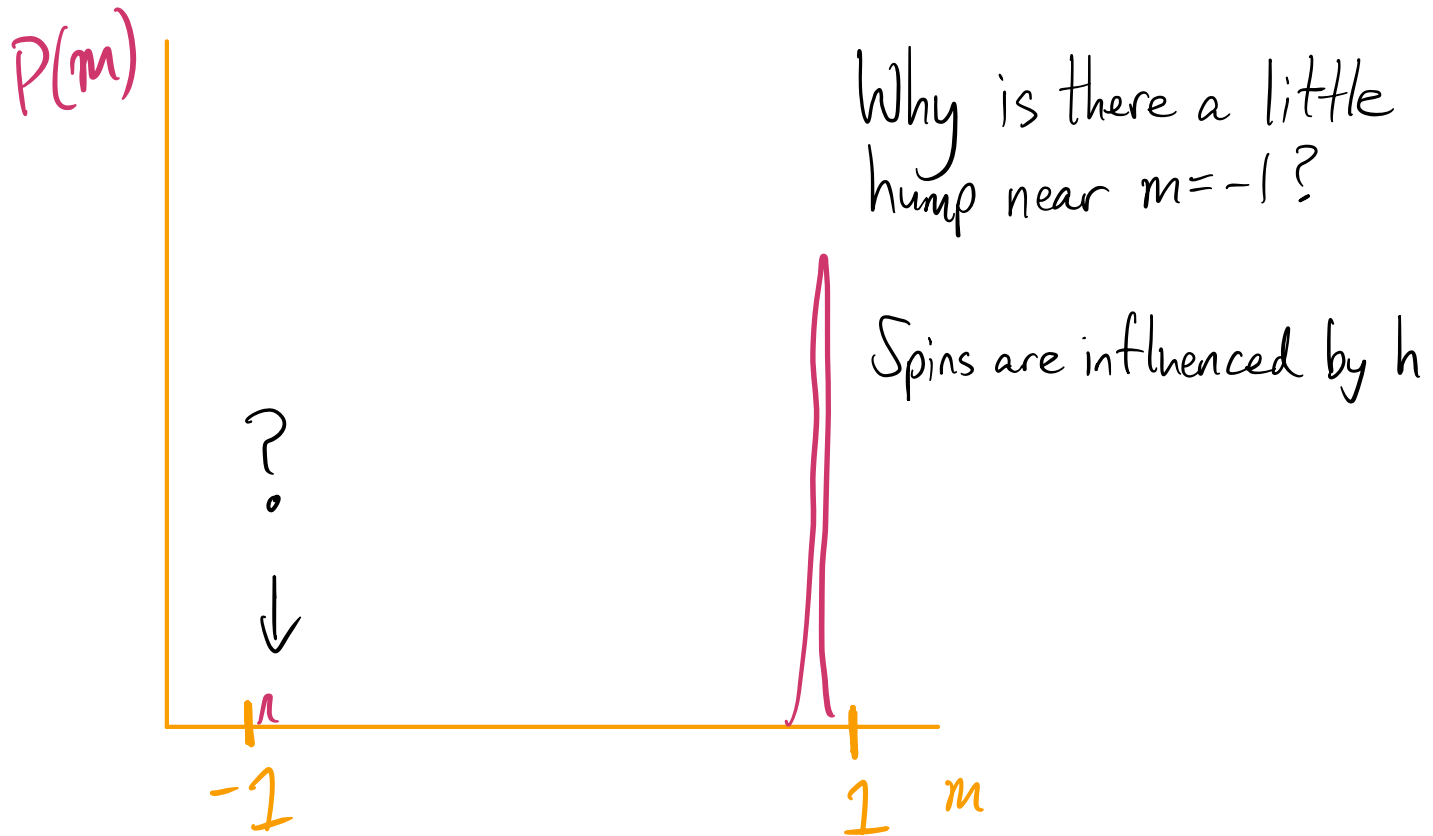
There is another way to study the model even when it is not analytically tractable. **Sampling**

How does $\langle m \rangle$ depend on h & T ?

(all possible v are in each hat, but all v do not have the same probability in each hat.)



Collect lots of samples from the orange hat and get the marginal distribution for m .



Marginal distributions $\longleftrightarrow$ Effective energies