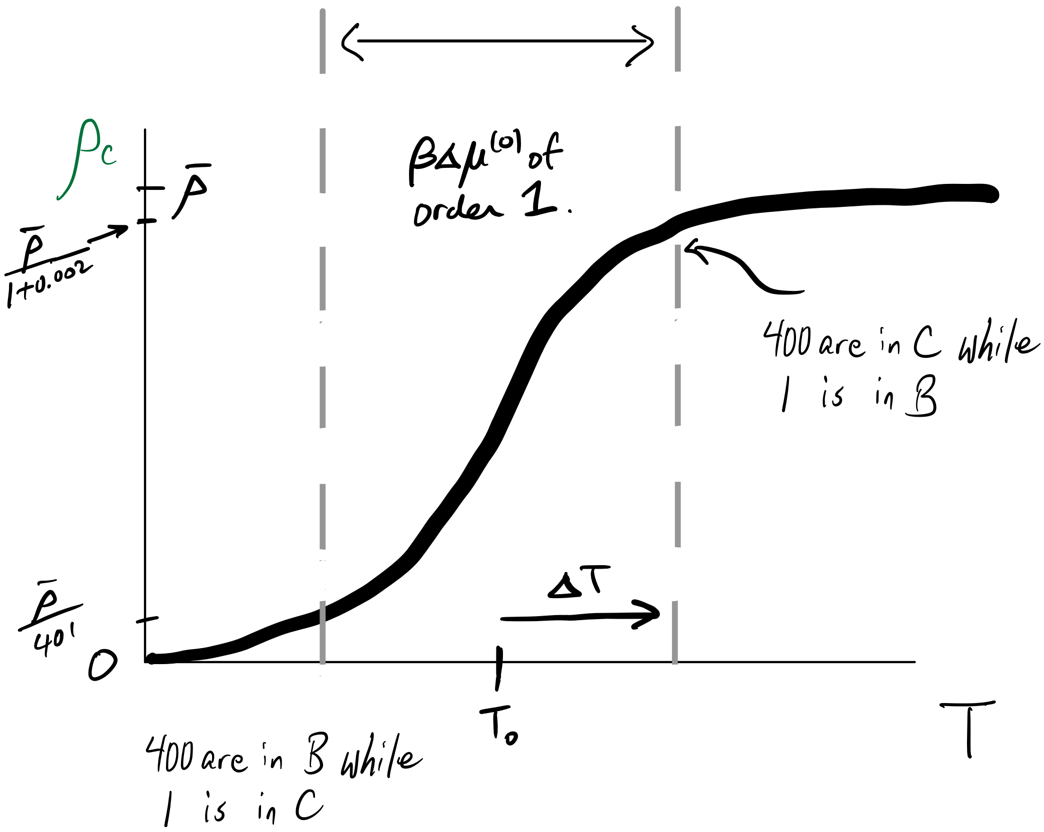
Lecture 15

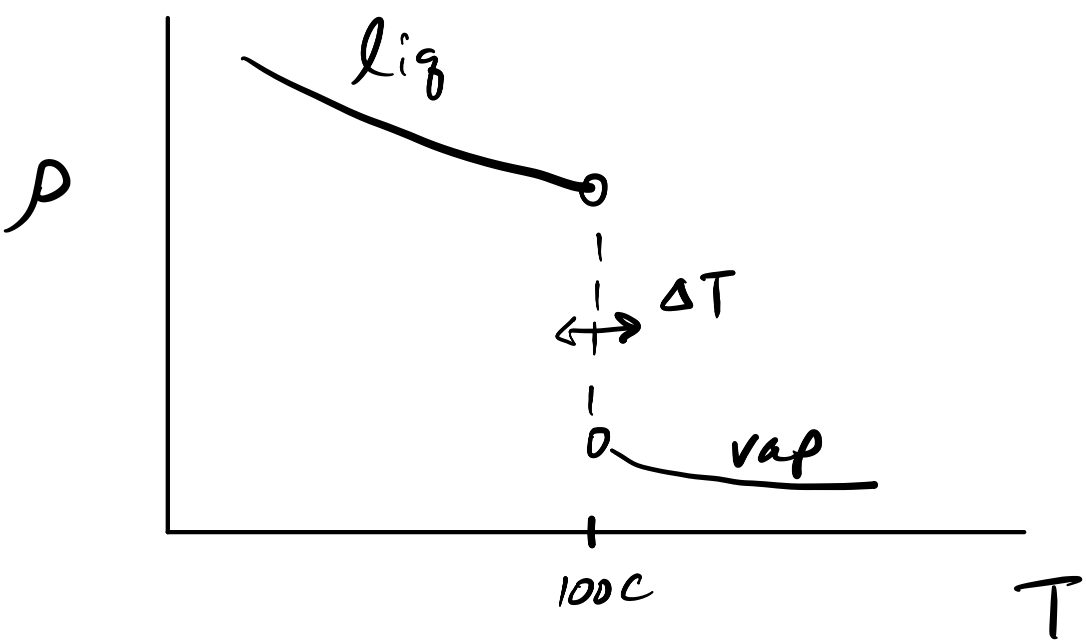
Recall from last lecture...

How broad is this crossover? $\Delta T \approx$ a few degrees C.



By contrast, for a liquid-vapor transition at fixed p,

How broad? $\Delta T \approx 0$ Not broad at all!

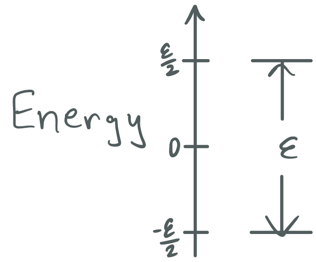


Phase transitions arise due to interactions.

Recall (Lecture 9)

Non-interacting spins in an external field.

$$\underset{s_1}{\uparrow}\ \underset{s_2}{\uparrow}\ \downarrow\ \uparrow\ \downarrow\ \downarrow\ \downarrow\ \cdots\ \underset{s_N}{\uparrow}$$



$$s_i = \begin{cases} +1, & \text{aligned with the field} \\ -1, & \text{anti-aligned} \end{cases}$$

$$\nu = \{s_1, s_2, \ldots, s_N\} \qquad E(\nu) = -\frac{\epsilon}{2}\sum_i s_i$$

We are after a relationship between T and the fraction of excited spins (an excited spin has $s_i = -1$)

$$p(s_1, s_2, \ldots, s_N) \propto e^{-\beta E(s_1, s_2, \ldots s_N)} = e^{\beta \frac{\epsilon}{2}\sum_i s_i}$$

It factorizes! $\Longrightarrow$

$$= e^{\beta\frac{\epsilon}{2}s_1}\, e^{\beta\frac{\epsilon}{2}s_2} \cdots e^{\beta\frac{\epsilon}{2}s_N}$$

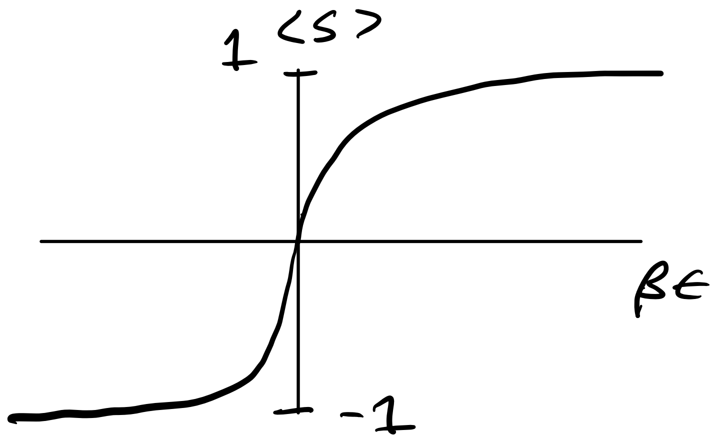
$$= p(s_1)\,p(s_2)\ldots p(s_N)$$

$\leftarrow$ spins are statistically independent (b/c they are non-interacting)

Using this, we found...

$$\langle s \rangle = p(s=1) * 1 + p(s=-1) * (-1)$$

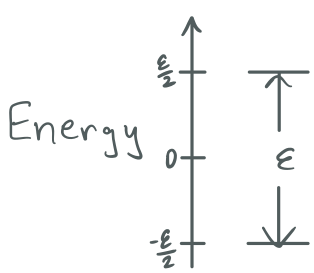
$$= \frac{e^{\beta\epsilon/2} - e^{-\beta\epsilon/2}}{e^{\beta\epsilon/2} + e^{-\beta\epsilon/2}} = \tanh\left(\frac{\beta\epsilon}{2}\right)$$



$$\Rightarrow \langle s \rangle = \frac{e^{\beta\epsilon} - 1}{e^{\beta\epsilon} + 1} \quad (*)$$

A smooth function of $\beta\epsilon$

How would you change ϵ in an experiment?



Tune the field strength H which would then change ϵ

$$E(\nu) = -\mu_0 H \sum_i S_i$$

$h = \mu_0 H$

magnetic moment

field strength

$$\underset{S_1}{\uparrow}\ \underset{S_2}{\uparrow}\ \downarrow\ \uparrow\ \downarrow\ \downarrow\ \downarrow\ \dots\ \underset{S_N}{\uparrow}$$

Actual spins tend to align with their neighbors (QM has a role here – exchange interaction)

Ising Hamiltonian

$$E(\nu) = \underbrace{-h\sum_i S_i}_{\text{non-interacting}} \underbrace{-\frac{J}{2}\sum_{i,j}{}' S_i S_j}_{\text{an interaction between spins}}$$

"Coupling constant"

If aligned, $S_i S_j = 1$

If not, $S_i S_j = -1$

The ½ allows us to include $i=1$, $j=2$ and $i=2$, $j=1$ terms in the sum – "double counting"

' is shorthand that means we only include nearest neighbor i + j.

Note

$$E\Big|_{h=0} = -\frac{J}{2}\sum_{i,j}' s_i s_j$$

has up-down symmetry

$$\underset{s_1}{\uparrow}\ \underset{s_2}{\uparrow}\ \downarrow\ \uparrow\ \downarrow\ \downarrow\ \downarrow\ \cdots\ \underset{s_N}{\uparrow}$$

$$E(\qquad) = E_{top}$$

$$E(\qquad) = E_{bottom} \quad \underset{s_1}{\downarrow}\ \underset{s_2}{\downarrow}\ \uparrow\ \downarrow\ \uparrow\ \uparrow\ \uparrow\ \cdots\ \underset{s_N}{\downarrow}$$

$$E_{top} = E_{bottom}$$

In thermal equilibrium, the probability of a microstate comes from the Boltzmann distribution (as always)

$$P(\nu) = \frac{e^{\beta h \sum_i s_i + \frac{\beta J}{2}\sum_{i,j}' s_i s_j}}{Q}$$

where

$$Q = \sum_\nu e^{\beta h \sum_i s_i + \frac{\beta J}{2}\sum_{i,j}' s_i s_j}$$

$$= \sum_{S_1=\pm1}\sum_{S_2=\pm1}\cdots\sum_{S_N=\pm1} e^{\beta h \sum_i S_i + \frac{\beta J}{2}\sum'_{i,j} S_i S_j}$$

When $J=0$, we're back to the non-interacting spins!

Factorized probability distribution $\qquad q = e^{-\beta E_{up}} + e^{-\beta E_{down}}$

$e^{-\beta h} + e^{\beta h}$

$$Q = \underbrace{\sum_{S_1=\pm1} e^{\beta h S_1}}_{q}\ \underbrace{\sum_{S_2=\pm1} e^{\beta h S_2}}_{q}\cdots\underbrace{\sum_{S_N=\pm1} e^{\beta h S_N}}_{q} = q^N$$

$$= \left(e^{-\beta h} + e^{\beta h}\right)^N$$

But when $J \neq 0$, things are not so easy.

The distribution doesn't factorize, so I must sum over the 2^N microstates, not just the microstates for a single spin.

We are still after a relationship between T and h and the fraction of ~~excited~~ up spins

Net Magnetization: $M = \sum_{\substack{\text{spins} \\ i}} S_i$

Net Magnetization per site: $m = \frac{M}{N}$

If we could explicitly compute $Q(\beta h, \beta J)$...

$$-\frac{\partial \ln Q}{\partial \beta} = \langle E \rangle \qquad \frac{\partial \ln Q}{\partial (\beta h)} = \langle M \rangle$$

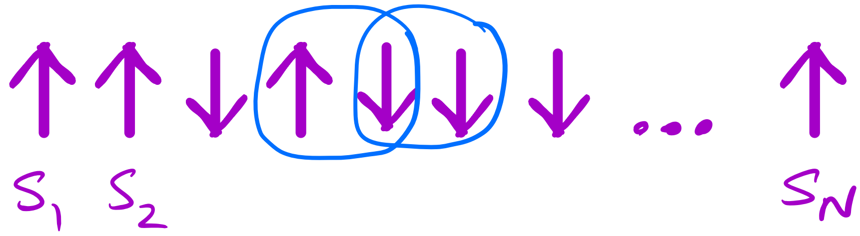
$$Q = \sum_{\nu} e^{\beta h \sum_i S_i + \frac{\beta J}{2} \sum'_{i,j} S_i S_j}$$

$$\frac{\partial \ln Q}{\partial (\beta h)} = \frac{1}{Q} \frac{\partial Q}{\partial (\beta h)} = \frac{1}{Q} \sum_{\nu} \underbrace{\left(\sum_i S_i \right)}_{M(\nu)} e^{\beta h \sum_i S_i + \frac{\beta J}{2} \sum'_{i,j} S_i S_j}$$

$$= \sum_{\nu} M(\nu) \, P(\nu) = \langle M \rangle$$

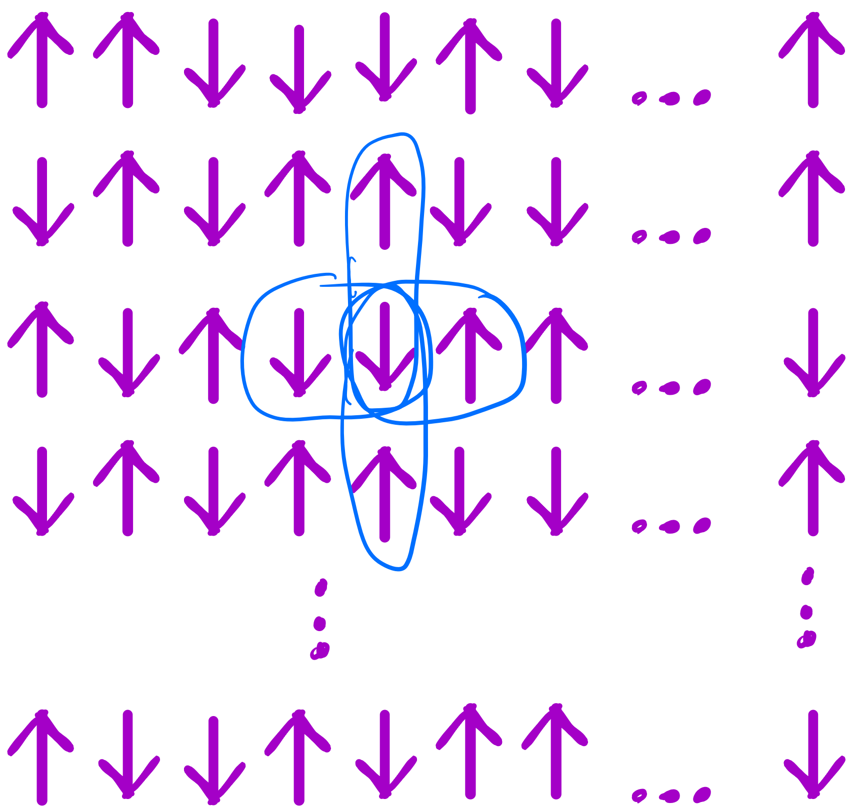
Well can we explicitly compute $Q(\beta h, \beta J)$? It depends.

1D ...



Yup. You can do it. Transfer matrices are fun!

2D ...

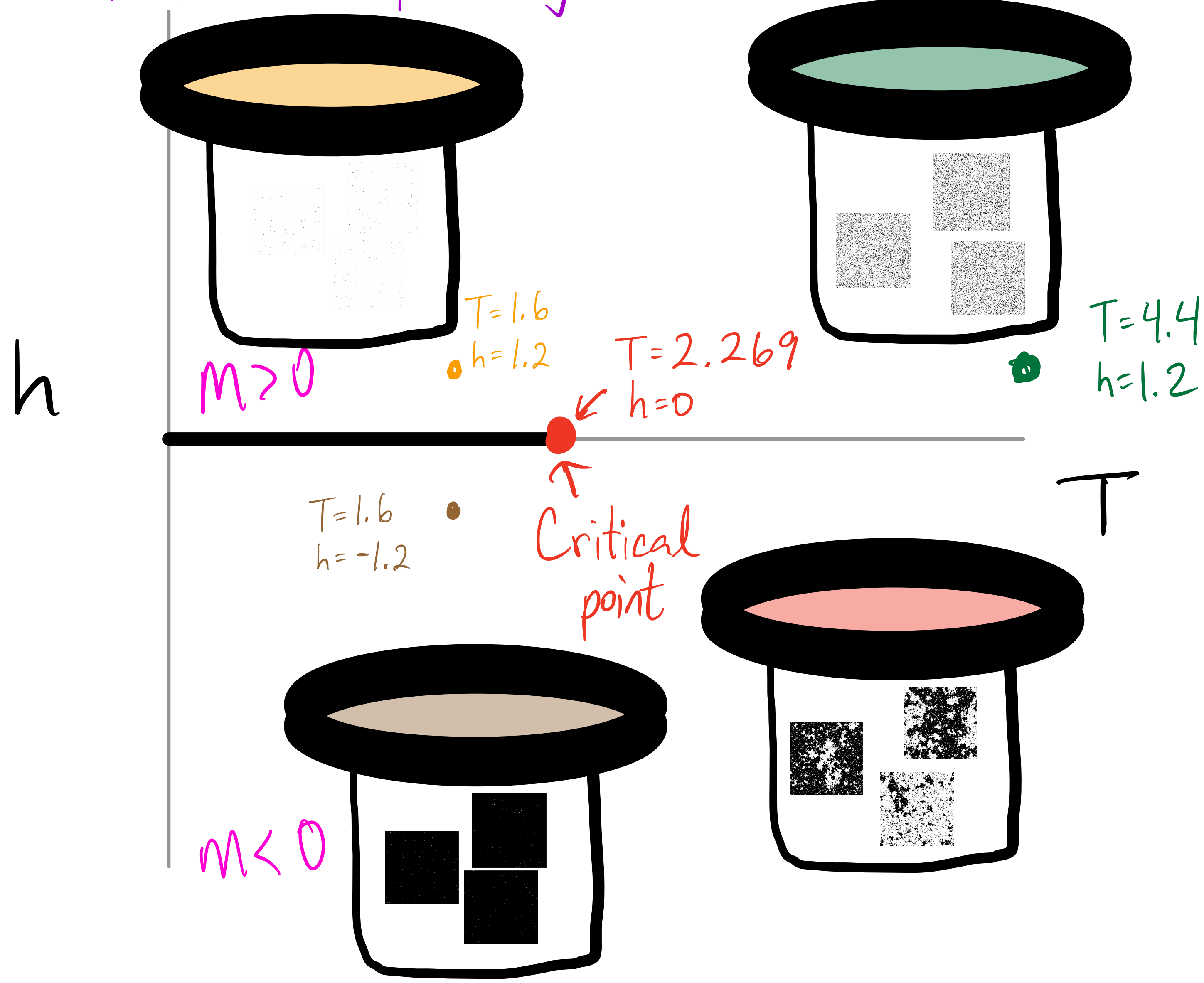


One can do it by hand. Perhaps not you or I. Lars Onsager did it.

$>$2D ... Nope

There is another way to study the model even when it is not analytically tractable. Sampling

How does $\langle m \rangle$ depend on h + T?
(all possible ν are in each hat, but all ν do not have the same probability in each hat.)



Collect lots of samples from the orange hat and get the marginal distribution for m.

Why is there a little hump near m=-1?

Spins are influenced by h

Marginal distributions $\Longleftrightarrow$ Effective energies