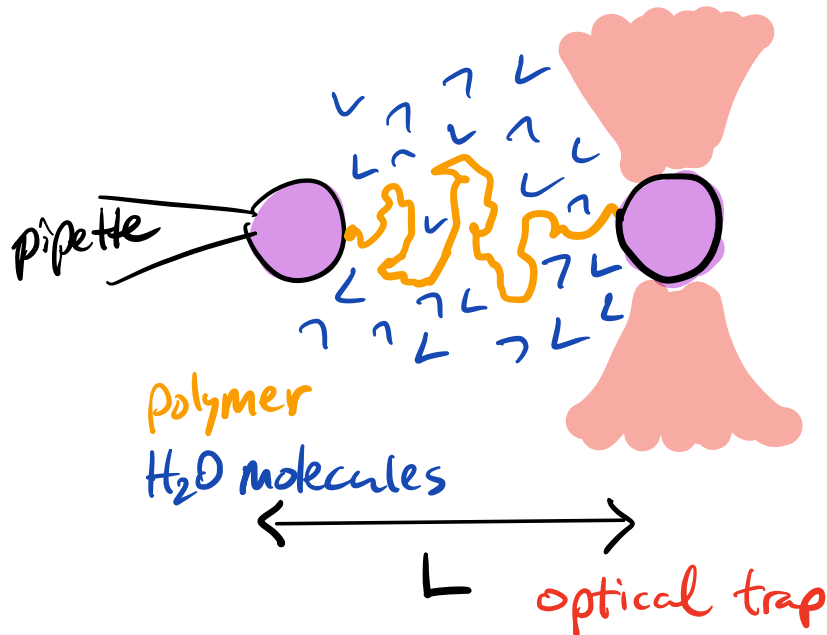


Lecture 14

Recall from last lecture ...



If I pull quickly, how much work do I do?

Usually $W > W_{rev}$, but not **always** true.
It depends on the trajectory's particular fluctuations,

on average \rightarrow

If I pull quickly, how much work do I do?

$$\langle W \rangle > W_{rev}$$

Suppose I want to measure a free energy difference. How do I do that?

- Perform a reversible process by perturbing $S \rightarrow L$ $0 \rightarrow W$ $L \rightarrow Y$.
Measure the work to execute that reversible process.

- $$\frac{P_F(W)}{P_R(L-W)} = e^{\beta(W-\Delta A)}$$

$$\Rightarrow \int dW P_F(W) e^{-\beta(W-\Delta A)} = \int dW P_R(L-W)$$

$$\langle e^{-\beta(W-\Delta A)} \rangle_F = 1$$

Jarzynski
Equality

Average over
trajectories being
stretched from
 0 to L

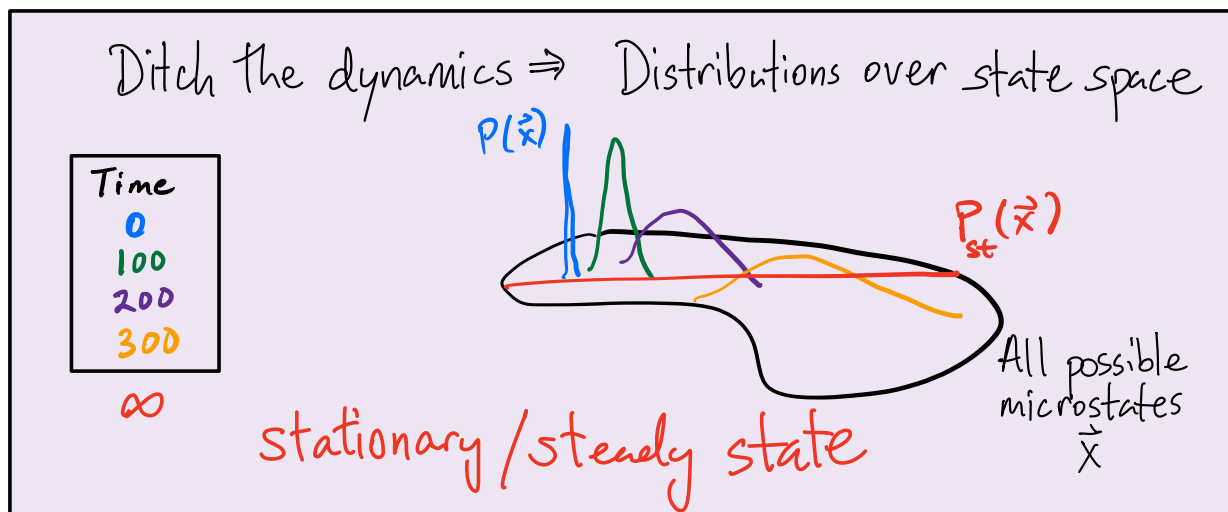
Will depend
on the trajectory

$$e^{\beta\Delta A} \langle e^{-\beta W} \rangle_F = 1$$

$$\Rightarrow \langle e^{-\beta W} \rangle_F = e^{-\beta\Delta A}$$

$$\Rightarrow k_B T \ln \langle e^{-\beta W} \rangle_F = \Delta A$$

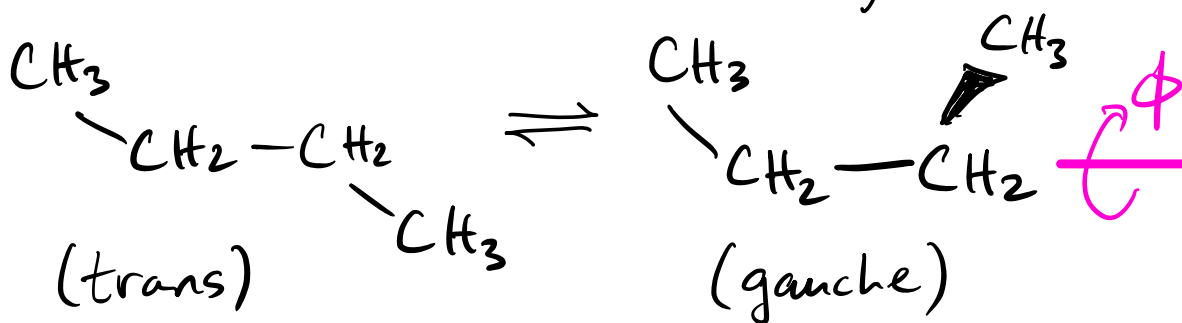
Remember that we started our study of statistical mechanics by discarding dynamics



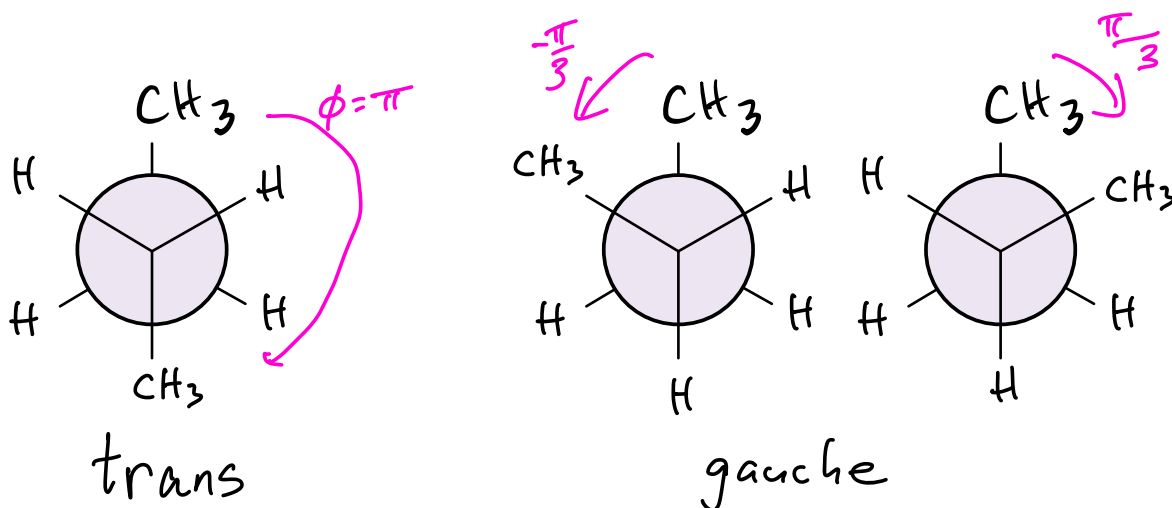
But if we are going to perturb faster than the system can relax then we cannot ditch the dynamics.

A crash course in a few aspects of dynamics of chemical systems, i.e. kinetics.

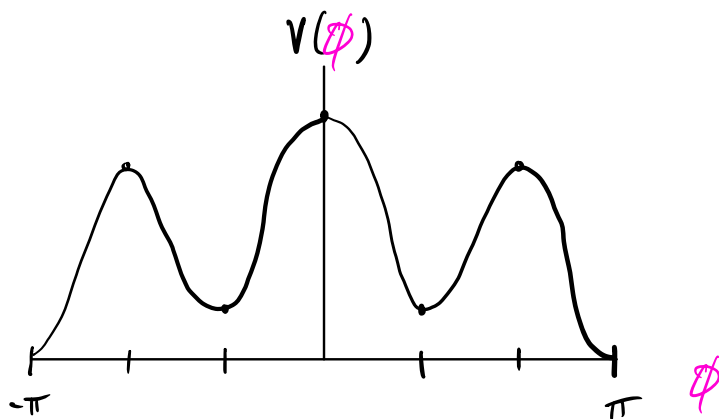
Let's consider chemical isomerization



ϕ : The angle between the two CH₃-CH₂ bonds



Imagine I gave you a function $V(\phi)$ for the energy as a function of this angle.



Every possible angle ϕ would occur with some probability (in equilibrium)

How does that $P(\phi)$ relate to $V(\phi)$?

Assuming the molecule can exchange energy with an outside world...

$$P(\phi) \propto e^{-\beta V(\phi)}$$

How does this energy $V(\phi)$ connect with dynamics?

Evidently the dynamics must converge to a steady state that visits ϕ with

$$P(\phi) = \sum_{\nu} P(\nu) \delta(\phi(\nu) - \phi) \propto e^{-\beta V(\phi)},$$

but in classical mechanics we have something more.

Hamilton's Eqs.:

$$\text{positions} \rightarrow \dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\text{momenta} \rightarrow \underbrace{\dot{p}}_{\substack{\uparrow \\ \text{acceleration}}} = - \frac{\partial H}{\partial q} = - \underbrace{V'(q)}_{\text{force}}$$

Here $q \leftrightarrow \phi$ $p \leftrightarrow$ an angular momentum for that ϕ

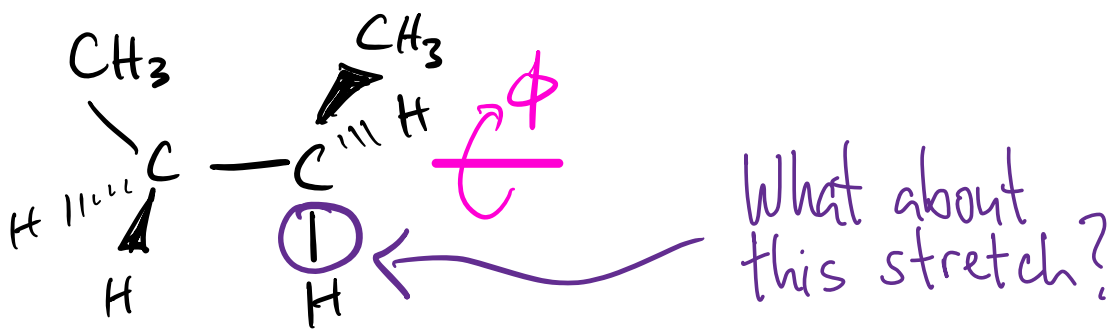
$$\dot{p} = -V'(q) \quad (\text{This is } F=ma)$$

An energy function tells you how you will move.

$$\left(\begin{array}{l} \text{Similarly in quantum mechanics ...} \\ |\Psi(t)\rangle = e^{-iHt/\hbar} |\Psi(0)\rangle \\ \sim \text{"Hamiltonian generates dynamics"} \end{array} \right)$$

But we must be careful with $V(\phi)$.
It is not a Hamiltonian. Why not...?

- No term with momenta (kinetic energy)
- The actual potential energy will be high dimensional, not only dependent on ϕ .



The energy depends on that too, but it is not captured if you only look at ϕ .

$V(\phi)$ is not a potential energy. It is actually a free energy and $V'(\phi)$ is an average force.

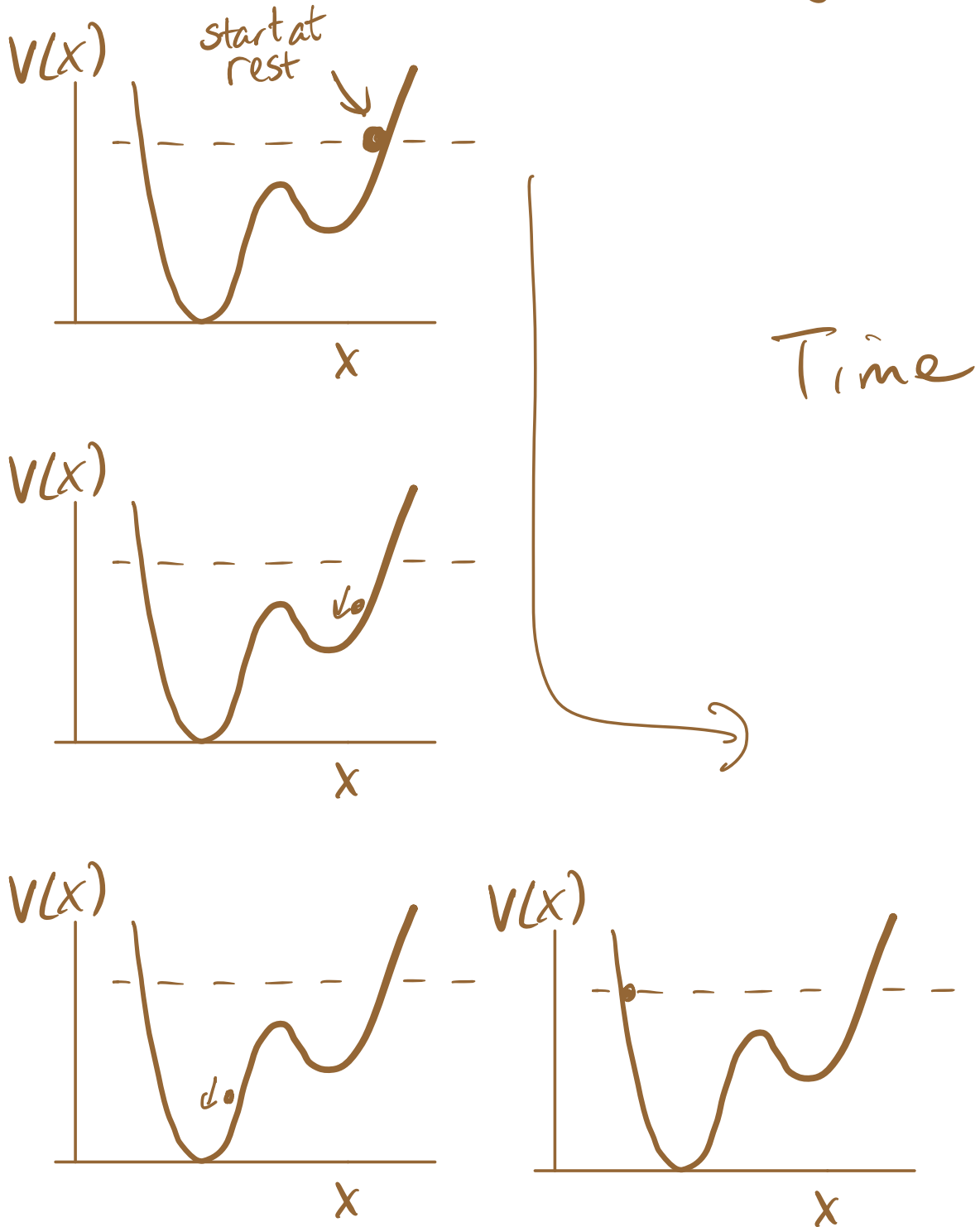
An average force? What am I averaging over?

The average force along the ϕ direction, averaging over all degrees of freedom we are not measuring.

Often $V(\phi)$ is called the potential of mean force.

Note: Degrees of freedom we're not measuring includes degrees of freedom of the electrons, so $V(\phi)$ even includes the average effects of quantum mechanics.

An actual potential energy:

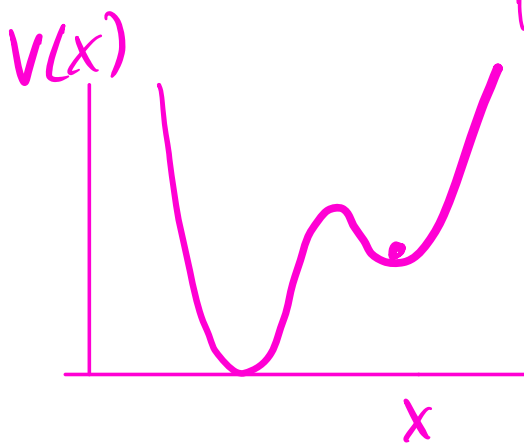


But if we were looking at a potential of mean force...



fast/common

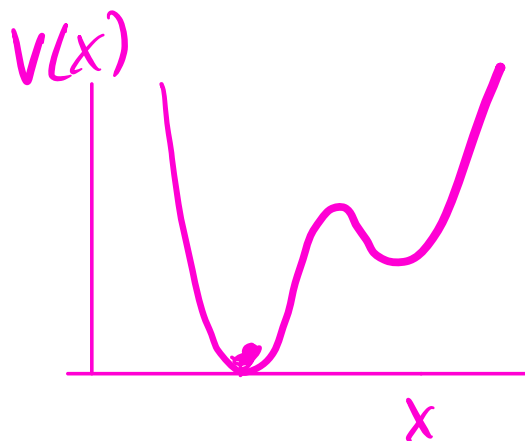
very rare



$$V(x) = -k_B T \ln P(x)$$

slow

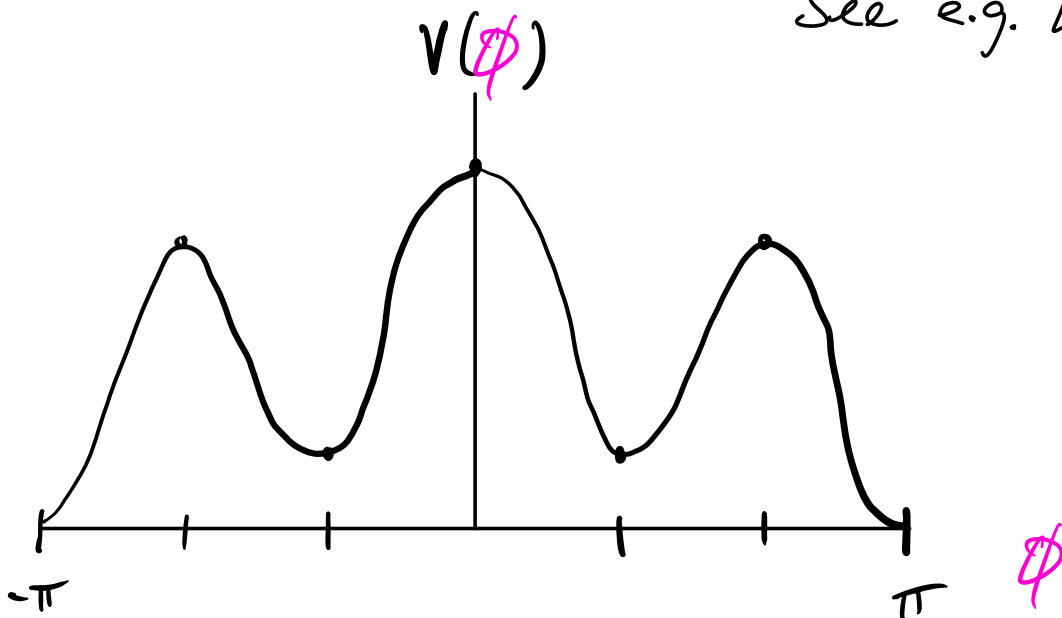
slow

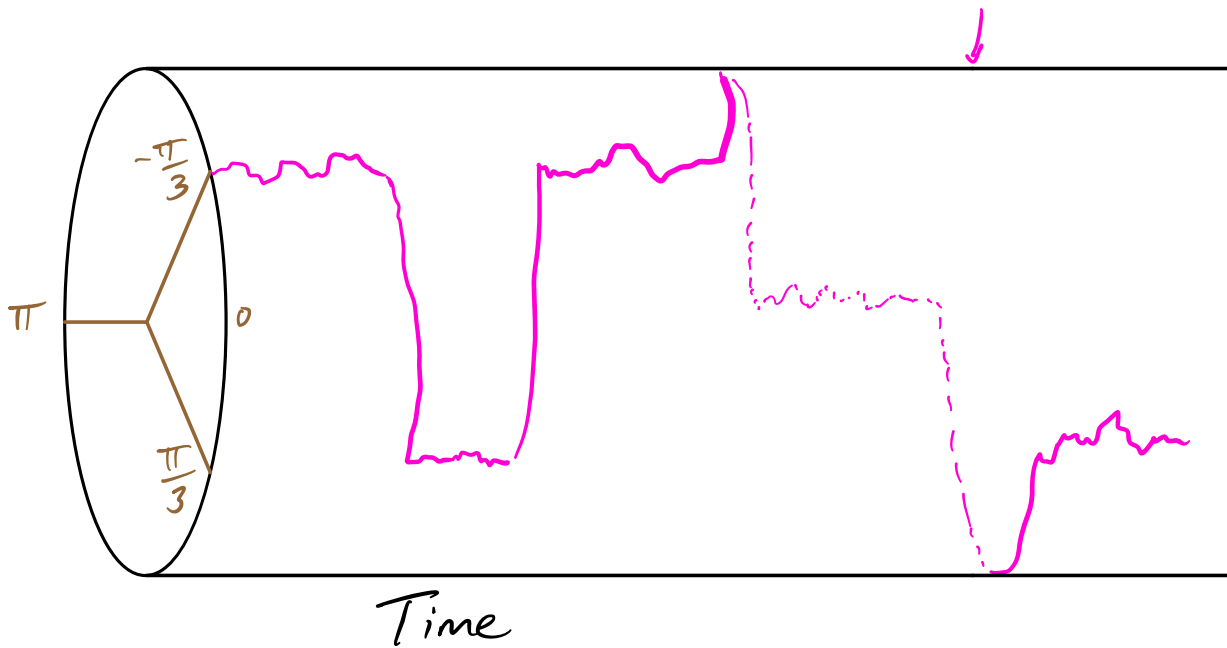


We can no longer describe motion of the *single degree of freedom* by energy-conserving equations of motion like Newton's laws.

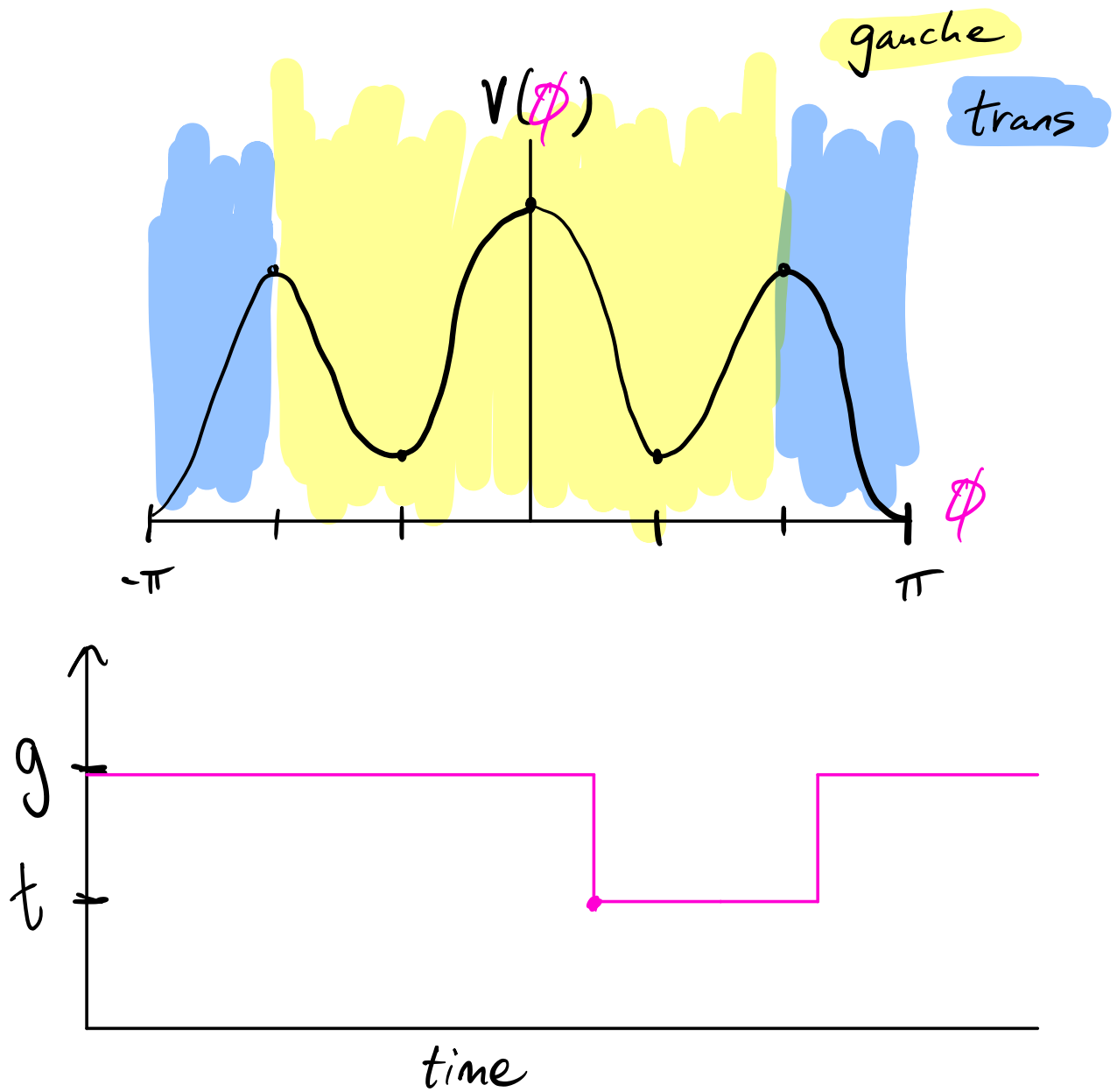
Instead we model the effect of the "integrated out" degrees of freedom as random perturbations to the *special degree of freedom*.

See e.g. Langevin



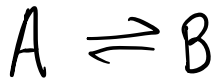


By specifying only ϕ , we are already clumping together microstates. Let's take this a step further...

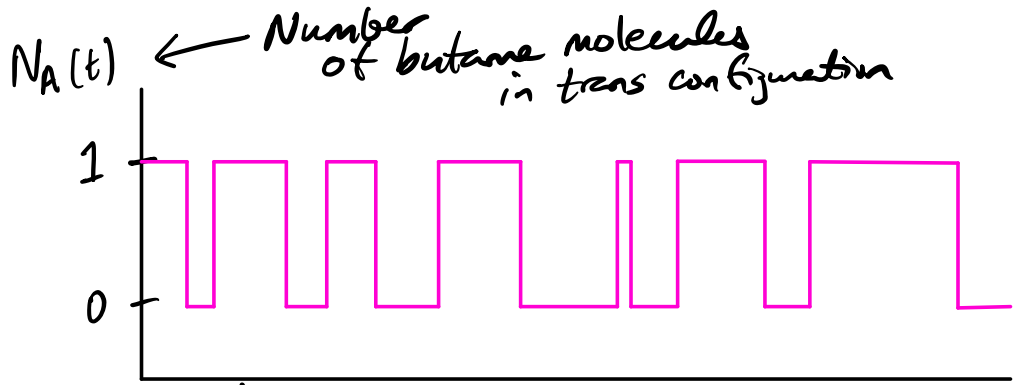


Let's develop a statistical model that can randomly generate trajectories for our

"coarse-grained two-state kinetics"

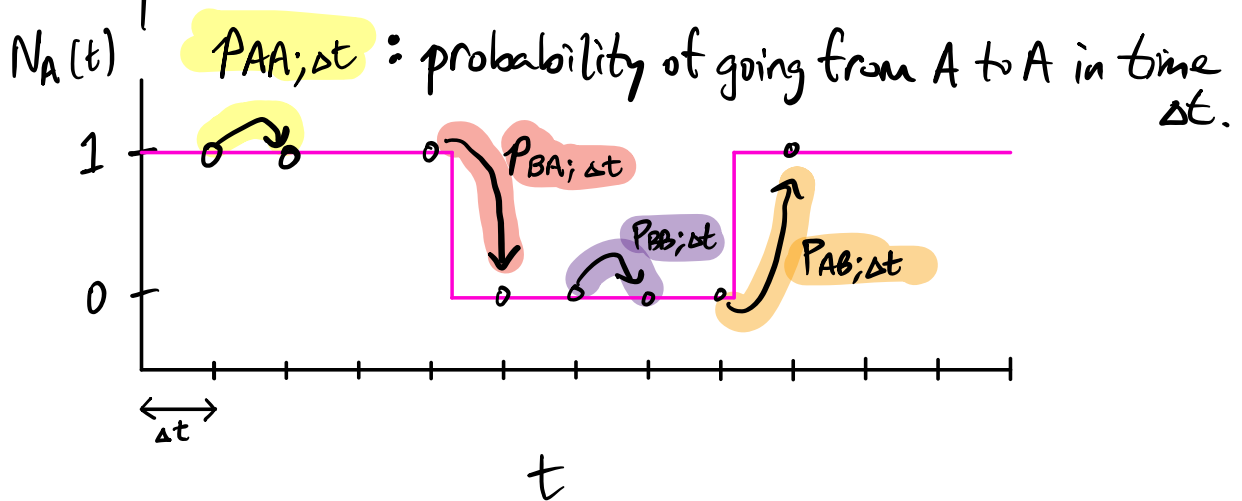


P_A : probability of A (trans)
 P_B : probability of B (gauche)



Time t

Zoom in



"Markov approximation: Only my current state impacts the probability of the next event. The past history has become irrelevant."

Start in A.

Probability of being in A at Δt ? $P_{AA;\Delta t}$

Probability of being in B at Δt ? $P_{BA;\Delta t}$

Probability of being in some state, A or B, at Δt ?

$$1 = P_{AA;\Delta t} + P_{BA;\Delta t}$$

(not all parameters
can be tuned independently)

Start in B...

Probability of being in A at Δt ? $P_{AB;\Delta t}$

Probability of being in B at Δt ? $P_{BB;\Delta t}$

Start in a mixed state:

Initial probability p_A of starting in A and
 p_B of starting in B.

$$\begin{aligned} P_A(\Delta t) &= \text{start in A \& stay there} + \text{start in B \& move to A} \\ &= p_A(0) P_{AA;\Delta t} + p_B(0) P_{AB;\Delta t} \end{aligned}$$

$$\begin{pmatrix} P_A(\Delta t) \\ P_B(\Delta t) \end{pmatrix} = \begin{pmatrix} P_{AA;\Delta t} & P_{AB;\Delta t} \\ P_{BA;\Delta t} & P_{BB;\Delta t} \end{pmatrix} \begin{pmatrix} P_A(0) \\ P_B(0) \end{pmatrix}$$

$$\overset{\bullet}{t} \xrightarrow{M} \overset{\bullet}{t+\Delta t}$$

What is the right Δt ?

Δt may be more of an artifact & you want to imagine being able to measure the state infinitely quickly, in which case $\Delta t \rightarrow 0$.

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} P_A(t) \\ P_B(t) \end{pmatrix} &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left[\begin{pmatrix} P_A(t+\Delta t) \\ P_B(t+\Delta t) \end{pmatrix} - \begin{pmatrix} P_A(t) \\ P_B(t) \end{pmatrix} \right] \\ &= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left[\begin{pmatrix} P_{AA;\Delta t} & P_{AB;\Delta t} \\ P_{BA;\Delta t} & P_{BB;\Delta t} \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} P_A(t) \\ P_B(t) \end{pmatrix} \\ &= \lim_{\Delta t \rightarrow 0} \begin{pmatrix} \frac{P_{AA;\Delta t} - 1}{\Delta t} & \frac{P_{AB;\Delta t}}{\Delta t} \\ \frac{P_{BA;\Delta t}}{\Delta t} & \frac{P_{BB;\Delta t} - 1}{\Delta t} \end{pmatrix} \begin{pmatrix} P_A(t) \\ P_B(t) \end{pmatrix} \end{aligned}$$

$$\lim_{\Delta t \rightarrow 0} \frac{P_{AB; \Delta t}}{\Delta t} = k_{AB} \quad \text{Rate}$$

Probability per unit time of transitioning from B to A in an infinitesimal moment of time.

$$\lim_{\Delta t \rightarrow 0} \frac{P_{BA; \Delta t}}{\Delta t} = k_{BA}$$

Diagonal matrix element?

$$\lim_{\Delta t \rightarrow 0} \frac{P_{AA; \Delta t} - 1}{\Delta t}$$

Remember

$$1 = P_{AA; \Delta t} + P_{BA; \Delta t}$$

$$= \lim_{\Delta t \rightarrow 0} \frac{(1 - P_{BA; \Delta t}) - 1}{\Delta t} = -k_{BA}$$

$$\frac{d}{dt} \begin{pmatrix} P_A(t) \\ P_B(t) \end{pmatrix} = \begin{pmatrix} -k_{BA} & k_{AB} \\ k_{BA} & -k_{AB} \end{pmatrix} \begin{pmatrix} P_A(t) \\ P_B(t) \end{pmatrix}$$

(“A continuous-time master equation”)