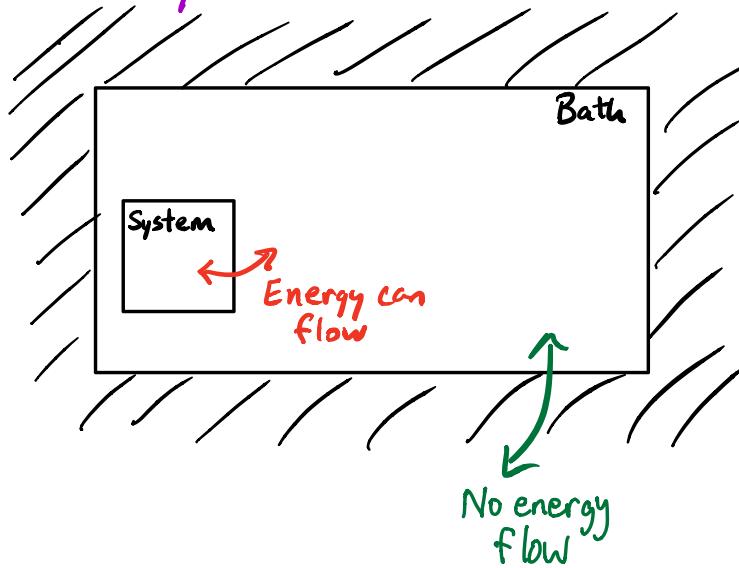


Lecture 5

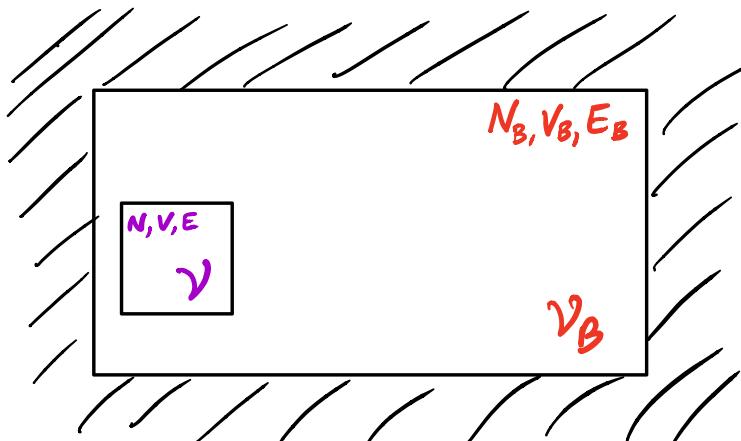
Recall from last lecture...

System + (Thermal) Bath



Although energy can flow between system and bath, they are isolated from the outside.

N_T, V_T, E_T



Variables with no flow:

$$N_T = N + N_B$$

$$V_T = V + V_B$$

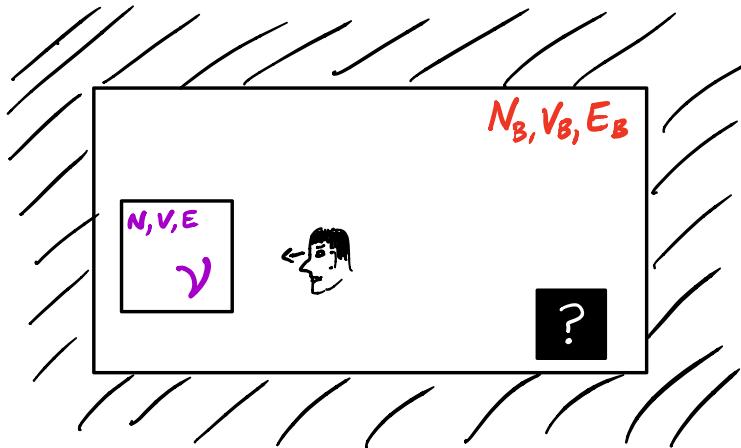
Variable with flow:

$$E_T = E(v) + E_B(v_B)$$

↓ ↑
 System Bath
 microstate microstate

Microstate of the full system + bath is defined by

$$\gamma = \{v, v_B\}$$



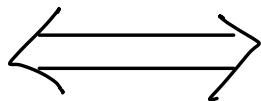
What if I cannot observe the state of the bath?

From the conservation of energy we still know something about the bath state merely by measuring v

If the system is in state v , then the bath could be in any state v_B for which:

$$E_B(v_B) = E_T - E(v)$$

Probability of
System microstate
 ν



How many bath
microstates have
energy

$$\Omega_B(E_T - E(\nu))$$

This is the distribution
for the thing we observe.
It's what we're after.

We could get to it
if we knew how to
count this.

How do we connect the left and right sides?

RHS:

$$\Omega_B(E_T - E(\nu)) = \# \text{ of Bath microstates w/ energy } E_T - E(\nu)$$

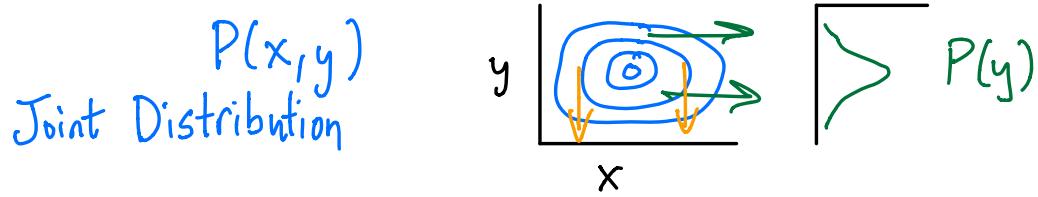
LHS:

$$P(\nu) = \sum_{\nu_B} P(\nu, \nu_B)$$

Marginal distribution

Joint distribution

Aside on Marginal Distributions



$$\underbrace{\int dx \int dy P(x,y)}_{P(x)} = 1 \quad (\text{Normalization of joint})$$

$$P(v) = \sum_{v_B} P(v, v_B) = \frac{1}{\mathcal{S}(E_T)} \mathcal{S}_B(E_T - E(v))$$

\curvearrowright Marginalization
over bath states

(1) What is $\mathcal{S}(E_T)$?

Why not write $\mathcal{S}(N_T, V_T, E_T)$?

(2) Why is $\mathcal{S}(E_T)$ the right normalization?

(3) Where does the $\mathcal{S}_B(E_T - E(v))$ come from?

(1) What is $\Omega(E_T)$?

Why not write $\Omega(N_T, V_T, E_T)$?

Lazy shorthand for $\Omega(N_T, V_T, E_T)$. Because we are not changing volumes or #'s of particles at the moment, we are focused only on the energy-dependence.

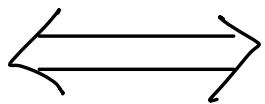
(2) Why is $\Omega(E_T)$ the right normalization?

$$P(v, v_B) = P(v_T) = \begin{cases} \frac{1}{\Omega(E_T)} & v + v_B \text{ s.t. } E(v) + E(v_B) = E_T \\ 0, & \text{otherwise} \end{cases}$$

(3) Where does the $\Omega_B(E_T - E(v))$ come from?

$$\sum_{v_B} P(v, v_B) = \frac{1}{\Omega(E_T)} \times \underbrace{\left(\begin{array}{l} \text{(How many } v_B \text{ satisfied)} \\ E(v) + E(v_B) = E_T \end{array} \right)}_{\Omega_B(E_T - E(v))}$$

Probability of
System microstate
 ν



How many bath
microstates have
energy $E_T - E(\nu)$

$$P(\nu) \propto \Omega_B(E_T - E(\nu))$$

Now we employ a (reasonable) approximation.
When the V_{bath} is ~~HUGE~~,

$$E(\nu) \ll E_T \dots$$

What is $\Omega_B(E_T - E)$ in the limit of
small E ? **Taylor Expand!**

$$f(x-x_0) \approx f(x) - x_0 f'(x) + \frac{x_0^2}{2} f''(x) + \dots$$

(Let's actually Taylor expand
 $\ln \Omega_B(E_T - E)$ not $\Omega_B(E_T - E)$)

Taylor expand $\ln \Omega_B(E_T - E)$ about $E=0$

Zeroth Order : $\ln \Omega_B(E_T - E) \Big|_{E=0}$

First Order : $E \left(\frac{\partial \ln \Omega_B}{\partial E} \right) \Big|_{E=0}$

How do the # of bath microstates change with a change in system energy E ?

Notice that $E_T - E = E_B \Rightarrow$

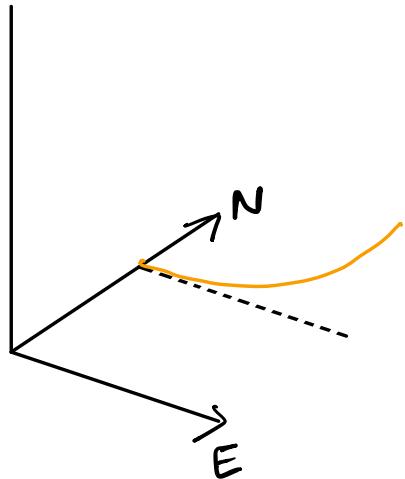
$$\left(\frac{\partial \ln \Omega_B}{\partial E} \right) \Big|_{E=0} = \frac{\partial \ln \Omega_B}{\partial E_B} \frac{\partial E_B}{\partial E} \quad \text{Chain Rule}$$

$$= - \left(\frac{\partial \ln \Omega_B}{\partial E_B} \right)_{N_B, V_B}$$

Well really we should have been writing

$$\Omega_B(N_B, V_B, E_B) \quad (\text{Not } \underline{\text{only}} \text{ a function of } E_B)$$

$$S_B(N, V, E)$$



slopes of $S_B(N, V, E)$
along a slice with N, V
fixed.

The BIG INSIGHT...

$$\left(\frac{\partial \ln \Omega_B}{\partial E_B} \right)_{N_B, V_B}$$

has nothing to do with the system.
It is a property of the bath!

Let's just define (for now) that bath property as

$$\beta = \left(\frac{\partial \ln \Omega_B}{\partial E_B} \right)_{N_B, V_B}$$

The Taylor expansion becomes

$$\ln \Omega_B(E_T - E) \approx \ln \Omega_B(E_T) - E \left(\frac{\partial \ln \Omega_B}{\partial E_B} \right)_{N_B, V_B} + \dots$$

$$\ln \Omega_B(E_T - E) \approx \ln \Omega_B(E_T) - \beta E + \dots$$

?

Exponentiating both sides yields

$$\underbrace{\Omega_B(E_T - E)}_{\propto e^{-\beta E} \propto P(\gamma)} = \Omega_B(E_T) e^{-\beta E}$$

Q: Why am I discarding $\Sigma_B(E_T)$?

A: It doesn't have to do with the system energy E , so it is just a constant that won't vary from one system microstate to the next.

By contrast, E is a function of ν .

Let's combine things...

Probability of
System microstate ν \longleftrightarrow How many bath
microstates have
energy $E_T - E(\nu)$

$$\underline{P(\nu)} \propto \underline{\Sigma_B(E_T - E(\nu))}$$

$$\underline{\Sigma_B(E_T - E)} \propto \underline{e^{-\beta E}}$$

$$P(\nu) \propto e^{-\beta E}$$

$$\boxed{\beta = \frac{1}{k_B T}}$$

Boltzmann Distribution.