1. **Three-state kinetics.** Consider a three-state kinetic scheme $1 \underset{k_1'}{\overset{k_1}{\rightleftharpoons}} 2 \underset{k_2'}{\overset{k_2}{\rightleftharpoons}} 3$. That is to say,

$$\frac{dp_1}{dt} = -k_1 p_1 + k_1' p_2$$

$$\frac{dp_2}{dt} = k_1 p_1 - k_1' p_2 + k_2 p_3 - k_2' p_2$$

$$\frac{dp_3}{dt} = -k_2 p_3 + k_2' p_2,$$

where the net probability is normalized, $p_1 + p_2 + p_3 = 1$, for all times t.

(i) The concentrations of species 1 and 3 will relax to their equilibrium concentrations with two superimposed exponential decays. Solve the system of differential equations to show that

$$p_1(t) = p_1^{(eq)} + x_1 e^{-t/\tau_1} + x_2 e^{-t/\tau_2}$$

$$p_3(t) = p_3^{(eq)} + y_1 e^{-t/\tau_1} + y_2 e^{-t/\tau_2}, \tag{1}$$

where τ_1 and τ_2 are the two relaxation times with $\tau_1 > \tau_2$. Note that $p_i^{(eq)}$ is the equilibrium or long time limit of $p_i(t)$. Express $p_1^{(eq)}, p_3^{(eq)}, \tau_1$ and τ_2, in terms of the four rate constants, k_1, k_1', k_2, and k_2'. To solve for x_1, x_2, y_1, and y_2 you would also need to match the initial conditions $p_1(0)$ and $p_3(0)$. These expressions are nastier, so I'm not asking for them here.

Hint: First use conservation of probability to eliminate p_2 from the problem, obtaining a system of two differential equations for $\dot{p}_1(t)$ and $\dot{p}_3(t)$ in terms of p_1 and p_3:

$$\begin{pmatrix} \dot{p}_1(t) \\ \dot{p}_3(t) \end{pmatrix} = M \begin{pmatrix} p_1(t) \\ p_3(t) \end{pmatrix} + b. \tag{2}$$

where M is a matrix and b is a vector, the components of which are all expressed in terms of the four rate constants in the problem. Remember that the solutions to this sort of differential equation are expressed as a superposition

$$\begin{pmatrix} p_1(t) \\ p_3(t) \end{pmatrix} = \begin{pmatrix} p_1^h(t) \\ p_3^h(t) \end{pmatrix} + \begin{pmatrix} p_1^p(t) \\ p_3^p(t) \end{pmatrix},$$

of a homogeneous solution solving

$$\begin{pmatrix} \dot{p}_1^h \\ \dot{p}_3^h \end{pmatrix} = M \begin{pmatrix} p_1^h \\ p_3^h \end{pmatrix}$$

and a particular solution solving

$$M \begin{pmatrix} p_1^p \\ p_3^p \end{pmatrix} + b = 0.$$

The homogeneous solution can be solved (up to constants of integration) by moving to an eigenbasis, and this should give you the τ's. The particular solution corresponds to the (time-independent) equilibrium probabilities, $p_i^{(eq)}$.

(ii) The conditions $k_1' \gg k_1$ and $k_2' \gg k_2$ coincide with state 2 being far less probable than both states 1 and 3. When these conditions hold, use the results from Part (ii) to determine $K_{\text{eq}} = p_3^{(\text{eq})}/p_1^{(\text{eq})}$, x_1, y_1 and τ_1 and show that x_2 and y_2 are comparatively negligible.

> Hint: Your expressions in (i) should involve expressions that have products of a primed and unprimed rate, e.g., $k_1' k_2$, as well as products of two unprimed rates, e.g., $k_1 k_2$. In the considered limit, $k_1' k_2 \gg k_1 k_2$, so you will often be able to neglect products of unprimed rates. In approximating τ_1 and τ_2, you will first want to approximate the eigenvalues of the matrix M. One of those eigenvalues could be written as
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> $$\lambda_1 = \frac{k_1 + k_1' + k_2 + k_2'}{2} \left[-1 + \sqrt{1 - 4\frac{k_1 k_2 + k_1' k_2 + k_1 k_2'}{(k_1 + k_1' + k_2 + k_2')^2}} \right]$$
>
> The second term in the square root will be small (why?), so you can Taylor expand the square root ($\sqrt{1-x} \approx 1-x/2$). Try to at least follow along enough to see why $\lambda_1 = \tau_1^{-1}$ will matter but $\lambda_2 = \tau_2^{-1}$ will not.

(iii) With the extreme inequalities of Part (ii) assumed at the outset, use a steady-state approximation to determine the time dependence of $p_1(t)$ and $p_3(t)$ and demonstrate this time dependence is the same as that found in part (ii).

2. **Do you smell that?** I'm worried people are going to fall asleep in lecture, so I decide to open up a bottle of perfume at the front of the room the moment I see someone start to nod off. Let's try to figure out how long it will take before they smell it.

   (i) Let us first assume that the random walk models we've worked so laboriously to understand can help us explain the situation! In other words, we assume that each perfume molecules can be modeled as moving in some direction until it collides with an air molecule, at which point the perfume randomly gets kicked in a new directly. We have seen that the spread of the random walkers (in this case perfume molecules) goes like $\langle \delta \boldsymbol{R}^2 \rangle_T = 2dDT$, where d is the dimensionality and D is the perfume's diffusion constant. Since we live in a three dimensional world, $\boldsymbol{R}^2 = X^2 + Y^2 + Z^2$ and $d = 3$. If perfume has a diffusion constant in air of 10^{-6} m^2s^{-1}, estimate the time for the perfume to reach the dozing student 5 meters away.

   (ii) Is your answer to (i) reasonable? What else could be going on?

   (iii) While we're crunching numbers, let's think about the *self-diffusion* of water, that is to say one molecule diffuses in a background of other water molecules (as opposed to the perfume in a background of air). The self-diffusion constant for a water molecule in liquid water is about $10^{-5} cm^2/sec$. Assuming the liquid is at equilibrium and not stirred, what are the typical times for a water molecule to move one molecular diameter (about 0.3 nm)? What about across the surface of a protein (about 1 nm)? Finally, a macroscopic distance (about 1 cm)?